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WHAT IN HIGH SCHOOL MATHEMATICS IS OF MOST IMPORTANCE AS A PREPARATION FOR ANALYTICAL GEOMETRY AND THE CALCULUS IN COLLEGE?

By Maurice J. Babb.

I asked this question of one of my sophomore classes and they all answered, "Algebra!!" Then I asked "What part of algebra?" and they answered, "Simplification of all kinds of expressions, notion of + and —, transposition, substitution of other expressions for 'unknowns,' radicals, use of fractional and negative exponents, binomial theorem, solution of equations in one and two unknowns (either of first and second degree), quadratic equations, *especially the theory* (this last very emphatic), expressions containing the logarithmic notation, graphs, and the language."

Trigonometry they believed was next. The most important parts to them were definitions of the sine, cosine, etc., and their elementary relations, expression of any part of a right triangle in terms of two other parts, circular measure, graphs, a few formulas, much work in identities, accurate interpolation and the language.

They were surprised how few geometric facts they used and how comparatively seldom they used them. The square on the hypotenuse loomed up large, then the area of a triangle and especially that of the equilateral triangle, the circumference and area of a circle, loci, volumes of different solids especially solids of revolution, relations of homologous lines, areas and volumes of similar figures. Some geometry of the circle, and a few definitions such as those of a plane, and of parallel lines. This knowledge they needed in a generalized form and they admitted that it was the training in logic rather than the facts that made geometry helpful.

In arithmetic, the use of the fundamental operations with accuracy and dispatch seemed to have been lost to some extent and the decimal point bobbed up in unexpected places.

A Little Physics for Applications in Mechanics.—Everywhere I found a lack of knowledge of the "Principle of Permanence or no Exception." They, like some Romans of old, were desirous for the "new thing" rather than its interpretation in the language of the old.

Now I am rather inclined to discount the teacher who is always complaining about the preparation of his pupils. He is generally a poor teacher himself. Boys and girls forget a great many things from day to day and more from year to year; but there are poor teachers in the high schools as well as in the colleges and in the "grades." I have personal knowledge of the following incidents: An eight-grade teacher this last spring started the elements of algebra as given in Hamilton's Arithmetic. The questions were like the following:

$$\begin{array}{r} +6x \\ -4x \\ +2x \\ \hline \end{array}$$

In explaining it she told her pupils to subtract all the numbers from the first. The next day, being a frank and conscientious teacher, she told them she had made a mistake and that they should have added all the numbers. Now what could you do with that class after all that? Forty out of forty-two failed on the examination for entrance to the high school. She said it was lack of preparation and got away with it.

In my classes in pedagogy of secondary mathematics I have frequently found a considerable portion of the class who did not know anything about the theory of quadratic equations. They were entirely innocent of any relation between the sum and product of the roots of a quadratic equation and the coefficients thereof. Nor did they know of any way of finding out the character of those roots except by completing the square or use of the formula, and only then when it came out even. One teacher said the reason she liked algebra was because the answer always came out even.

In geometry many bright and experienced teachers told me a plane was shaped much like the samples Wanamaker sends out of his dress goods, and one teacher who has now turned to the ministry (a reverse process from that of some years ago), having drawn two cones side by side on the board, stated that one of them was a right cone, and in answer to further inquiry explained that the other was a left cone. These are all classroom experiences with successful practicing teachers presumably much above the average, and I could tell you some more.

I believe that the most important thing in high school mathematics or any other subject is a more thorough and intimate acquaintance with at least the elementary parts of the subjects to be pursued in college. This should be gotten by intelligent study from well-written texts and under the direction and criticism of a well-trained and skillful teacher. If this standard cannot be obtained at once it should be approached as rapidly as possible.

Then the teacher should realize where to draw the line, where to place the high lights, where to merely emphasize by drill, where to give some or all an inkling of what it is all about. The magazines long ago understood the psychology of "continued in our next." But be it remembered that mathematics for the teacher and mathematics for the boy are very different things. The teacher wants to know all the boy might ask; the boy wants to discuss all he sees the present necessity of, and can be stimulated by occasional forecasts of the future. The teacher should be able to understand every method the boy may try. Many a budding genius is crushed by uniformity of method. There is also some food for thought in the fact that Greek mathematics developed out of the attempt to solve the three impossible problems of antiquity. How like the old struggle against fate! "There is no royal road to geometry," and the doctrine of interest does not necessarily include the doctrine of ease. Knowledge is power and there is nothing effeminate about it.

All mathematicians know that a nomenclature is chosen after careful trial, because it is perspicuous to the eye and elastic enough to suit the purpose. We generally give the boy the nomenclature first and when he wants to solve a problem he tries all the formulas he can think of. A formula is useful only when it is not needed. A symbolism is good only when it is a crying necessity and a pleasure to the user.

There is a mathematical language. An eminent English clergyman named Sydney Jordan had a son who went up to matriculate. His father was much afraid of his examination in Euclid and told Jordan, Jr., to telegraph as soon as his fate was known. The young hopeful sent the following telegram: "Hymn 342, verse 5, last three lines." The father got out the hymn-book and read these words:

Sorrow vanquished,
Labor ended,
Jordan passed.

It was in language he could understand. Now many a bright mind could work mathematics if he only understood the symbolism. Any sensible boy believes the truth of many propositions in geometry, but it often takes the best minds a long while to discover that the analytic part of the proof has been omitted and that the theorem has been polished by the wise men of the ages as lovingly and delicately as Clark ever polished an object glass for a telescope, until it got its present definite shape. These theorems have been proved and improved. Some of the most elementary propositions like the preface need to be written last.

Another thing that affords difficulty is that the pupil starts to work the problem before he has examined all the conditions. A Sunday-school teacher was striving to make Noah a real man to his boys. He asked them what they thought Noah did all the time in the ark. One boy suggested that he might have been fishing. The teacher said possibly he might. Another boy, who had been thinking quietly, said he didn't think he could have fished long, " 'cause he only had *two* worms."

Because of the difficulty to translate into algebraic language it is almost impossible to get a result to a reading problem on a college examination paper unless it is a well-known variation to some stock problem and done by the aid of pure memory. Ordinary writing is wonderful to the fellow who doesn't know how. Mathematical language is as difficult as shorthand to the girl who doesn't have the key to the symbolism. Yet girls learn shorthand in a comparatively short time.

Many a wealthy man is unable to relish the dainties set before him and longs for the poor man's appetite. If they have the hunger they can be fed. A few years ago I had a boy in my sophomore class of civil engineers. Six years before that time he landed in New York an ignorant and almost penniless Yiddish lad who couldn't write his name in his own language and who understood no English. He said the only way he could be a gentleman was to be a scholar. At the time he was in my class he owned enough real estate to pay his dues in the building and loan, his university fees and for his board and clothing. What we all need is a good appetite and then proper food. But I do not believe in a predigested diet for a healthy mind. A bright boy cannot thrive on the same rations as his duller classmate and will, like Oliver Twist, rebel against the board.

The French teachers of mathematics are mathematicians, and their bright pupils are three years ahead of ours at the end of the high school course, while in Germany the entire class is at least two years ahead, both in subject matter and thoroughness. As to the reasons advanced for this evident superiority abroad in England, France and Germany, I refer you to the various magazine articles on the Perry movement, Klein's work in Germany and the more recent work in France. I would advise you to use these new ideas, as I would advise the boy with his problems: study the conditions carefully before you go to work.

Now I am going to tell you about some troubles we have in analytical geometry and calculus at the University of Pennsylvania. Transpositions offer great difficulty on account of the artificial way they are taught and the fact that the right-hand side of an equation is supposed forbidden ground for the unknown. Why not

$$70-3x=14-x+6x,$$
$$56=8x,$$
$$x=7.$$

Also

$$x/24-5x/42=17/36-11x/56$$

gives

$$x/12-5x/21=17/18-11x/28.$$

Now

$$12=2\times 2\times 3,$$
$$21=3\times 7,$$
$$18=2\times 3\times 3,$$
$$28=2\times 2\times 7.$$

Therefore multiply by $2 \times 2 \times 3 \times 3 \times 7$, etc.

Never multiply out till you are forced to, but keep in the factored form

$$\frac{1}{1-x}+\frac{6}{x}=\frac{6-5x}{1-x},$$

$$\frac{6}{x}=\frac{5-5x}{1-x},$$

$$\frac{6}{x}=5,$$

$$\frac{6}{5}=x.$$

Birds of a feather flock together in mathematics. Very few boys seem to know the definition of a root of an equation and would have been satisfied with all the answers that generous algebra would give.

Proportion has been sadly neglected of late.

$$\frac{1+z}{1-z}=\frac{n+1}{n},$$

$$\frac{2z}{2}=\frac{1}{2n+1}.$$

division and composition

$$z=\frac{1}{2n+1}.$$

Also solve for n.

Solve for x in the following:

(*a*) $$\sqrt{x^2-x+2}=z-x,$$

(*b*) $$\sqrt{2+x-x^2}=(2-x)z.$$

In radicals

$$\sqrt{6}\times\sqrt{10}=2\sqrt{15},$$

immediately on account of the common $\sqrt{2}$. This also is applicable to imaginaries where we take out the required number of common factors including the $\sqrt{-1}$.

Thus

$$\sqrt{-8}\times\sqrt{-6}=\sqrt{-2\times4}\times\sqrt{-2\times3}$$

$$=-2\times2\sqrt{3}=-4\sqrt{3}.$$

Every factor which has a partner can emerge, and those that have no duplicate have to stay where they were.

Very few boys examine their results in problems like the following:

$$\sqrt{x-3}-\sqrt{2x-5}=\sqrt{x-6},$$

$$\sqrt{x}\pm\sqrt{y}=\pm\sqrt{a}.$$

How difficult it is to isolate the x's and y's:

$$Ax+By+C=0,$$

$$\frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1,$$

or

$$y=-\frac{A}{B}x-\frac{C}{B},$$

or from

$$b^2x^2+a^2y^2=a^2b^2$$

obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and

$$y=\pm\frac{b}{a}\sqrt{a^2-x}.$$

The simplification of

$$(x-ae)^2+y^2=e^2\left(x-\frac{a}{e}\right)^2.$$

$$e>1 \text{ and } e<1$$

offers great difficulties because the boys cannot pick out the coefficients of the different powers of x.

Of course anybody can do almost any problem if he lives long enough, but what we want is the correct answer in a reasonable time and with simple examples that means "instanter."

$$a^2(1-e^2)=b^2,$$

$$e=\frac{\sqrt{a^2-b^2}}{a}.$$

Now construct *e, a/e* and *ae,* having given *a* and *b.*

Eliminate *K* from

$$x=-\frac{ma^2k}{m^2a^2+b^2},$$

$$y=-\frac{b^2k}{m^2a^2+b^2}.$$

Elimination is not always done by addition and subtraction. The function idea and the definition of a root are closely allied. We can substitute root for proof.

Evaluate

$$3x^2-6x+4y^2+4y-5=0,$$

when

$$x=1,\ 2,\ x_1+1,$$

$$y=-2,\ 2,\ y_1-\tfrac{1}{2}.$$

Factoring and finding roots are two variations of the same thing.

The factor theorem is very important, but factoring often is overdone, as only the simplest cases are really needed.

Simplify $\dfrac{x^4-2x^3+7x+14}{2x^3-4x^2+6x-12}.$

Improper fractions are tabooed in elementary calculus. Thus:

$$\frac{2x^2-3}{3x^2-7x}=\frac{2}{3}+\frac{14x-9}{3x(3x-7)}.$$

From theory of quadratics we have, if

$$\begin{cases} y^2=4ax, \\ x=sy+t, \end{cases}$$

$$y^2-4asy-4at=0,$$

therefore

$$\frac{y_1 + y_2}{2} = 2as.$$

and the diameter of a parabola is parallel to the axis.

Similarly if

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad y = mx + k,$$

then, when roots are equal,

$$y = mx \pm \sqrt{a^2m^2 - b^2},$$

and we have the equation of a tangent with a given direction or slope.

Eliminating y in this same system of equations we obtain

$$(a^2m^2 - b^2)x^2 + 2mka^2x + a^2(k^2 + b^2) = 0.$$

Our tangent at infinity, viz., the asymptote, is given when the two roots are infinite, that is when the coefficients of the highest powers in succession $= 0$; whence

$$m = \pm b/a, \ k = 0,$$

hence

$y = \pm b/a \cdot x$ is equation of asymptote.

This theory of the quadratic is in all the elementary algebras, and, like all solutions of single or simultaneous equations is much clearer when illustrated by graphs.

In trigonometry much facility is gained by having the pupil translate directly from the figure without going through the geometry as a medium. Much of the trouble is a matter of notation

$$\cos(\sin^{-1}n)$$

means if the sine of an angle is n what is its cosine. The projection of a line equals its length multiplied by the cosine of the angle between the line and its projection.

In geometry it is very important that the student be familiar with the solids and areas of revolution and their manner of generation.

Suppose an isosceles triangle inscribed in a circle of radius v. Let its altitude be y and its base $2x$, then $y(2v - y) = x^2$ from the geometry of the circle and we have either the base or altitude in terms of the other. Now try the similar problem for the circumscribed isosceles triangle, and the inscribed rectangle. Then revolve and obtain the sphere with its inscribed and circumscribed cone and inscribed cylinder and obtain the relations between the radius of the base and the altitude in terms of any given volume or surface.

Most of the processes of algebra are very artificial and uninteresting until illuminated by the calculus. Some day we may do as the French and Germans do, teach a good deal of the calculus and analytic geometry in the high school, but this will not properly come until long after our teachers will have illuminated the high school mathematics through their knowledge of its most powerful and interesting applications, the analytic geometry and the calculus. Let us hope for more power, more knowledge.

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